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World-Volume Action of the M Theory Five-Brane[1]

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Abstract

This paper presents a 6d world-volume action that describes the dynamics of the M theory five-brane in a flat 11d space-time background. The world-volume action has global 11d super-Poincaré invariance, as well as 6d general coordinate invariance and kappa symmetry, which are realized as local symmetries. The paper mostly considers a formulation in which general coordinate invariance is not manifest in one direction. However, it also describes briefly an alternative formulation, due to Pasti, Sorokin, and Tonin, in which general coordinate invariance is manifest. The latter approach requires auxiliary fields and new gauge invariances.

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1 Introduction

World-volume actions of p-branes encode much information about their dynamics. In the case of strings (in flat backgrounds) the world-volume theory has been quantized and used to construct the string perturbation expansion. In the case of p-branes with $p > 1$, one does not expect that it is possible to do the same. Still, many recent works have shown that an understanding of p-branes, including their excitations, can be very useful. Much non-perturbative information has been gleaned by considering vacua containing various branes of infinite extension. (A good example is provided by the 7-branes of F-theory [1].) Also, non-perturbative excitations described by wrapping p-branes about various cycles have played a central role in recent studies of black hole entropy as well as other problems [2, 3, 4]. We suspect that a more detailed characterization of p-brane world-volume dynamics will enable these studies to go further.

The actions for the class of supersymmetric p-branes whose only degrees of freedom are the superspace coordinates X and θ of the ambient space-time were constructed during the decade of the 1980's [5, 6, 7, 8, 9]. Much more recently, the actions for D-branes in type II theories have been constructed [10, 11, 12, 13]. In addition to the X and θ variables, these world-volume theories contain a U(1) gauge field with Born–Infeld self interactions [14, 15, 16, 17, 18, 19]. For maximally supersymmetric theories, the only significant p-brane action that remains to be formulated is that of the M theory five-brane [20, 21, 22, 23, 24]. This paper presents the solution.

The new feature that makes the M theory five-brane example somewhat more challenging than the other ones is the presence of a second-rank tensor gauge field, in addition to the X and θ coordinates [25]. This gauge field describes a chiral boson in the world volume, since its field strength is self-dual in the linearized approximation. It has been known for a long time that there is no straightforward way to construct a covariant action that describes propagation of the self-dual part of this field without also bringing in the anti-self-dual part [26]. Various proposals for dealing with this problem have been suggested over the years. The main one that we adopt is based on a formulation in which general coordinate invariance is only manifest in five of the six dimensions [27, 28, 29, 30]. It is also present in the sixth direction, but the transformation formulas that describe the symmetry are rather complicated. The bosonic part of the five-brane theory, constructed by this method, has been presented recently [31]. Another approach to the problem of the chiral boson uses an infinite number of auxiliary fields [32, 33, 34].

Very recently, a manifestly covariant formulation involving only a finite number of auxiliary fields (and compensating gauge invariances) has been introduced by Pasti, Sorokin, and Tonin [35, 36]. Constructions using the PST formulation turn out to be about as complicated as those in the formulation without manifest covariance. In fact, one of the new gauge invariances of the PST formulation involves the same subtleties as those of general coordinate invariance in the non-covariant approach, since one can gauge fix the PST formulas to obtain the non-covariant ones and show that compensating gauge transformations are the origin of the complicated general coordinate transformation.

Besides general coordinate invariance, the other essential symmetry of the world-volume theory of any super p-brane is a fermionic symmetry called kappa symmetry. It is always needed to remove half the degrees of freedom carried by the θ variables, leaving altogether eight propagating fermionic degrees of freedom. This is the same as the number of bosonic degrees of freedom, of course, as required by supersymmetry. The way this is achieved is by adding a suitable Wess–Zumino term to the action.

In all previous super p-brane examples, the global super-Poincaré symmetry (induced from an ambient flat space-time background) is implemented separately for the Wess–Zumino term and the other terms. The story in the case of the M theory five-brane has a surprising new feature. Namely, extending the bosonic five-brane theory to achieve global 11d super-Poincaré symmetry uniquely determines the complete action, including the Wess–Zumino term. The formula obtained in this way is then shown to have general coordinate invariance and local kappa symmetry. In the covariant PST formulation one is forced to organize the terms somewhat differently, so in that approach the story looks somewhat more conventional. Specifically, the covariant action divides naturally into two pieces: one piece is the supersymmetrized bosonic theory and the second is a separately supersymmetric Wess–Zumino term. The reason these statements are not in contradiction is that the PST gauge invariances, which are needed to achieve the right *bosonic* degrees of freedom, require that both terms be included.

This paper is organized as follows. Section 2 reviews the construction of the bosonic part of the M theory five-brane action in both the non-covariant and the PST formulations. Section 3 then describes the supersymmetrization of this theory and the determination of the Wess–Zumino term in the non-covariant formulation. The proof that the resulting theory has (non-manifest) general coordinate invariance is given in Section 4. Section 5 presents the proof of kappa symmetry. The verification of two crucial identities is relegated to a pair of appendices. This section also sketches the corresponding formulas in the PST formulation. Section 6 describes double dimensional reduction, which gives rise to a 4-brane in 10d spacetime. The resulting theory gives a dual formulation of the D4-brane of type IIA theory in which the theory is expressed in terms of a two-form gauge field instead of the dual U(1) vector gauge field. Some concluding remarks are made in Section 7.

2 Review of the Bosonic Theory

2.1 Formulation Without Manifest Covariance

Ref. [31] analyzed the problem of coupling a 6d self-dual tensor gauge field to a metric field so as to achieve general coordinate invariance. It presented a formulation in which one direction is treated differently from the other five. At the time that work was done, the author knew of no straightforward way to make the general covariance manifest. However, shortly thereafter a paper appeared [35] that presents equivalent results using a manifestly covariant formulation [36], which we refer to as the PST formulation. The relation between the two approaches will be described in the next subsection. As one might expect, they entail similar complications and there does not appear to be much advantage to one approach over the other. Therefore, we will present the supersymmetric M theory 5-brane action in the formulation without manifest covariance. This action corresponds to a partially gauge-fixed version of the corresponding action in the PST formulation.

In the present work we denote the 6d (world volume) coordinates by $\sigma^{\hat{\mu}} = (\sigma^\mu, \sigma^5)$, where $\mu = 0, 1, 2, 3, 4$. (In ref. [31] they were called $x^{\hat{\mu}}$.) The σ^5 direction is singled out as the one that will be treated differently from the other five.[2] The 6d metric $G_{\hat{\mu}\hat{\nu}}$ contains 5d pieces $G_{\mu\nu}, G_{\mu 5}$, and G_{55}. All formulas will be written with manifest 5d general coordinate invariance. As in refs. [30, 31], we represent the self-dual tensor gauge field by a 5×5 antisymmetric tensor $B_{\mu\nu}$, and its 5d curl by $H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]}$. A useful quantity is the dual

$$\tilde{H}^{\mu\nu} = \frac{1}{6}\epsilon^{\mu\nu\rho\lambda\sigma} H_{\rho\lambda\sigma}. \tag{1}$$

[2]This is a space-like direction, but one could also choose a time-like one. (See the discussion in sect. 2.2.) The reason we prefer this choice is that in section 6, where we perform a double dimension reduction to obtain a 4-brane in 10d, elimination of the special dimension leaves manifestly covariant equations.

It was shown in ref. [31] that a class of generally covariant bosonic theories could be represented in the form $L = L_1 + L_2 + L_3$, where[3]

$$
\begin{aligned}
L_1 &= -\frac{1}{2}\sqrt{-G} f(z_1, z_2), \\
L_2 &= -\frac{1}{4}\tilde{H}^{\mu\nu}\partial_5 B_{\mu\nu}, \\
L_3 &= \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{G^{55}}\tilde{H}^{\mu\nu}\tilde{H}^{\lambda\sigma}.
\end{aligned}
\tag{2}
$$

The notation is as follows: G is the 6d determinant ($G = \det G_{\hat{\mu}\hat{\nu}}$) and G_5 is the 5d determinant ($G_5 = \det G_{\mu\nu}$), while G^{55} and $G^{5\rho}$ are components of the inverse 6d metric $G^{\hat{\mu}\hat{\nu}}$. The ϵ symbols are purely numerical with $\epsilon^{01234} = 1$ and $\epsilon^{\mu\nu\rho\lambda\sigma} = -\epsilon_{\mu\nu\rho\lambda\sigma}$. A useful relation is $G_5 = GG^{55}$. The z variables are defined to be

$$
\begin{aligned}
z_1 &= \frac{\mathrm{tr}(G\tilde{H}G\tilde{H})}{2(-G_5)} \\
z_2 &= \frac{\mathrm{tr}(G\tilde{H}G\tilde{H}G\tilde{H}G\tilde{H})}{4(-G_5)^2}.
\end{aligned}
\tag{3}
$$

The trace only involves 5d indices:

$$
\mathrm{tr}(G\tilde{H}G\tilde{H}) = G_{\mu\nu}\tilde{H}^{\nu\rho}G_{\rho\lambda}\tilde{H}^{\lambda\mu}. \tag{4}
$$

The quantities z_1 and z_2 are scalars under 5d general coordinate transformations.

Infinitesimal parameters of general coordinate transformations are denoted $\xi^{\hat{\mu}} = (\xi^\mu, \xi)$. Since 5d general coordinate invariance is manifest, we focus on the ξ transformations only. The metric transforms in the standard way

$$
\delta_\xi G_{\hat{\mu}\hat{\nu}} = \xi\partial_5 G_{\hat{\mu}\hat{\nu}} + \partial_{\hat{\mu}}\xi G_{5\hat{\nu}} + \partial_{\hat{\nu}}\xi G_{\hat{\mu}5}. \tag{5}
$$

The variation of $B_{\mu\nu}$ is given by a more complicated rule, whose origin is explained in ref. [31]:

$$
\delta_\xi B_{\mu\nu} = \xi K_{\mu\nu}, \tag{6}
$$

where

$$
K_{\mu\nu} = 2\frac{\partial(L_1 + L_3)}{\partial\tilde{H}^{\mu\nu}} = K^{(1)}_{\mu\nu} f_1 + K^{(2)}_{\mu\nu} f_2 + K^{(\epsilon)}_{\mu\nu} \tag{7}
$$

[3]The formula given in ref. [31] has been rescaled by an overall factor of $-1/2$.

with

$$
\begin{aligned}
K^{(1)}_{\mu\nu} &= \frac{\sqrt{-G}}{(-G_5)}(G\tilde{H}G)_{\mu\nu} \\
K^{(2)}_{\mu\nu} &= \frac{\sqrt{-G}}{(-G_5)^2}(G\tilde{H}G\tilde{H}G\tilde{H}G)_{\mu\nu} \\
K^{(\epsilon)}_{\mu\nu} &= \epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{2G^{55}}\tilde{H}^{\lambda\sigma},
\end{aligned}
\tag{8}
$$

and we have defined

$$
f_i = \frac{\partial f}{\partial z_i}, \quad i = 1, 2. \tag{9}
$$

Assembling the results given above, ref. [31] showed that the required general coordinate transformation symmetry is achieved if, and only if, the function f satisfies the nonlinear partial differential equation [37]

$$
f_1^2 + z_1 f_1 f_2 + (\frac{1}{2}z_1^2 - z_2)f_2^2 = 1. \tag{10}
$$

As discussed in [30], this equation has many solutions, but the one of relevance to the M theory five-brane is

$$
f = 2\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2}. \tag{11}
$$

For this choice L_1 can reexpressed in the Born–Infeld form

$$
L_1 = -\sqrt{-\det\Big(G_{\hat{\mu}\hat{\nu}} + iG_{\hat{\mu}\rho}G_{\hat{\nu}\lambda}\tilde{H}^{\rho\lambda}/\sqrt{-G_5}\Big)}. \tag{12}
$$

This expression is real, despite the factor of i, because it is an even function of $\tilde{H}$. Eliminating the factor of i would correspond to replacing z_1 by $-z_1$, which also solves the differential equation. However, it is essential for the five-brane application that the phases be chosen as shown.

2.2 The PST Formulation

In ref. [35] (using techniques developed in ref. [36]) results equivalent to those of the preceding subsection are described in a manifestly covariant way. To do this, the field $B_{\mu\nu}$ is extended to $B_{\hat{\mu}\hat{\nu}}$ with field strength $H_{\hat{\mu}\hat{\nu}\hat{\rho}}$. In addition, an auxiliary scalar field a is introduced. The PST formulation has new gauge symmetries (described below) that allow one to choose the gauge $B_{\mu 5} = 0$, $a = \sigma^5$ (and hence $\partial_{\hat{\mu}} a = \delta^5_{\hat{\mu}}$). In this gauge, the covariant PST formulas reduce to those of sect. 2.1.

As will become clear, the scalar filed a is really a zero-form potential with one-form field strength da. Only the field strength needs to be single-valued. Furthermore, for the action to be nonsingular, it is necessary that the 6 manifold M_6 admit nowhere null closed one-forms and that da be restricted to the class of such one-forms. It is allowed to be either time-like or space-like, however. This topological restriction on M_6 is consistent with the conclusions reached in ref. [22]

Equation (12) expressed L_1 in terms of the determinant of the 6×6 matrix

$$M_{\hat{\mu}\hat{\nu}} = G_{\hat{\mu}\hat{\nu}} + i \frac{G_{\hat{\mu}\rho} G_{\hat{\nu}\lambda}}{\sqrt{-GG^{55}}} \tilde{H}^{\rho\lambda}. \tag{13}$$

In the PST approach this is extended to the manifestly covariant form

$$M^{\text{cov.}}_{\hat{\mu}\hat{\nu}} = G_{\hat{\mu}\hat{\nu}} + i \frac{G_{\hat{\mu}\hat{\rho}} G_{\hat{\nu}\hat{\lambda}}}{\sqrt{-G(\partial a)^2}} \tilde{H}^{\hat{\rho}\hat{\lambda}}_{\text{cov.}}. \tag{14}$$

The quantity

$$(\partial a)^2 = G^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}} a \partial_{\hat{\nu}} a \tag{15}$$

reduces to G^{55} upon setting $\partial_{\hat{\mu}} a = \delta^5_{\hat{\mu}}$, and

$$\tilde{H}^{\hat{\rho}\hat{\lambda}}_{\text{cov.}} \equiv \frac{1}{6} \epsilon^{\hat{\rho}\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\sigma}\hat{\tau}} H_{\hat{\mu}\hat{\nu}\hat{\sigma}} \partial_{\hat{\tau}} a \tag{16}$$

reduces to $\tilde{H}^{\rho\lambda}$. Thus $M^{\text{cov.}}_{\hat{\mu}\hat{\nu}}$ replaces $M_{\hat{\mu}\hat{\nu}}$ in L_1. Furthermore, the expression

$$L' = -\frac{1}{4(\partial a)^2} \tilde{H}^{\hat{\mu}\hat{\nu}}_{\text{cov.}} H_{\hat{\mu}\hat{\nu}\hat{\rho}} G^{\hat{\rho}\hat{\lambda}} \partial_{\hat{\lambda}} a, \tag{17}$$

which transforms under general coordinate transformations as a scalar density, reduces to $L_2 + L_3$ upon gauge fixing. It is interesting that L_2 and L_3 are unified in this formulation.

Let us now describe the new gauge symmetries of ref. [35]. Since degrees of freedom a and $B_{\mu 5}$ have been added, corresponding gauge symmetries are required. One of them is

$$\delta B_{\hat{\mu}\hat{\nu}} = 2\phi_{[\hat{\mu}} \partial_{\hat{\nu}]} a, \tag{18}$$

where $\phi_{\hat{\mu}}$ are infinitesimal parameters, and the other fields do not vary. In terms of differential forms, this implies $\delta H = d\phi da$. $\tilde{H}^{\hat{\rho}\hat{\lambda}}_{\text{cov.}}$ is invariant under this transformation, since it corresponds to the dual of Hda, but $dada = 0$. Thus the covariant version of L_1 is invariant under this transformation. The variation of L', on the other hand, is a total derivative.

The second local symmetry involves an infinitesimal scalar parameter φ. The transformation rules are $\delta G_{\hat{\mu}\hat{\nu}} = 0, \delta a = \varphi$, and

$$\delta B_{\hat{\mu}\hat{\nu}} = \frac{1}{(\partial a)^2} \varphi H_{\hat{\mu}\hat{\nu}\hat{\rho}} G^{\hat{\rho}\hat{\lambda}} \partial_{\hat{\lambda}} a + \varphi V_{\hat{\mu}\hat{\nu}}, \tag{19}$$

where the quantity $V_{\hat{\mu}\hat{\nu}}$ is to be determined. This transformation is just as complicated as the non-manifest general coordinate transformation in the non-covariant formalism. Rather than derive it from scratch, let's see what is required to agree with the previous formulas after gauge fixing. In other words, we fix the gauge $\partial_{\hat{\mu}} a = \delta^5_{\hat{\mu}}$ and $B_{\mu 5} = 0$, and figure out what the resulting ξ transformations are. We need

$$\delta a = \varphi + \xi \partial_5 a = \varphi + \xi = 0, \tag{20}$$

which tells us that $\varphi = -\xi$. Then

$$\begin{aligned} \delta_\xi B_{\mu\nu} &= \frac{1}{(\partial a)^2} \varphi H_{\mu\nu\hat{\rho}} G^{\hat{\rho}\hat{\lambda}} \partial_{\hat{\lambda}} a + \varphi V_{\mu\nu} + \xi H_{5\mu\nu} \\ &= -\xi \left(\frac{G^{\rho 5}}{G^{55}} H_{\mu\nu\rho} + V_{\mu\nu} \right) = \xi (K^{(\epsilon)}_{\mu\nu} - V_{\mu\nu}). \end{aligned} \tag{21}$$

Thus, comparing with eqs. (6) and (7), we need the covariant definition

$$V_{\hat{\mu}\hat{\nu}} = -2 \frac{\partial L_1}{\partial \tilde{H}^{\hat{\mu}\hat{\nu}}_{\text{cov.}}} \tag{22}$$

to achieve agreement with our previous results.

To summarize, we have learned that the covariant PST formulation has new gauge transformations, and one of them encodes the complications that end up in general coordinate invariance after gauge fixing. Thus this formalism is not simpler than the non-covariant one. However, it is more symmetrical, and it does raise new questions, such as whether there are other gauge choices that are worth exploring.

3 Supersymmetrization

The super–Poincaré symmetry of the flat 11d space-time background should be implemented as a global symmetry of the five-brane theory. In terms of superspace coordinates X^M and θ, the 11d supersymmetry transformation is given by

$$\delta\theta = \epsilon \quad \text{and} \quad \delta X^M = \bar{\epsilon} \Gamma^M \theta. \tag{23}$$

Our convention is that the index M takes the values $M = 0, 1, \ldots, 9, 11$. Skipping $M = 10$ may seem a bit peculiar, but then X^{11} is the 11th dimension. Also, the Dirac matrix $\Gamma_{11} = \Gamma_0 \Gamma_1 \ldots \Gamma_9$, which appears in ten dimensions as a chirality operator, is precisely the matrix we associate with the 11th dimension. The spinors ϵ and θ are 32-component Majorana spinors. The Dirac algebra is

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}, \tag{24}$$

where η_{MN} is the 11d Lorentz metric with signature $(- + + \ldots +)$.

As in other supersymmetric p-brane theories, two supersymmetric quantities are $\partial_{\hat{\mu}}\theta$ and

$$\Pi^M_{\hat{\mu}} = \partial_{\hat{\mu}} X^M - \bar{\theta}\Gamma^M \partial_{\hat{\mu}}\theta. \tag{25}$$

The appropriate choice for the world-volume metric is then the supersymmetric quantity

$$G_{\hat{\mu}\hat{\nu}} = \eta_{MN} \Pi^M_{\hat{\mu}} \Pi^N_{\hat{\nu}}. \tag{26}$$

Taking θ and X^M to be scalars under world-volume general coordinate transformations, $G_{\hat{\mu}\hat{\nu}}$ transforms in the standard way.

In addition, we require an appropriate supersymmetric extension of $H = dB$, which we write as

$$\mathcal{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - b_{\mu\nu\rho}, \tag{27}$$

or, in terms of differential forms, $\mathcal{H} = H - b_3$. The idea is to choose a b_3 whose supersymmetry variation is exact, so that it can be cancelled by an appropriate variation of B. The appropriate choice turns out to be

$$b_3 = \frac{1}{6} b_{\mu\nu\rho} d\sigma^\mu d\sigma^\nu d\sigma^\rho = \frac{1}{2}\bar{\theta}\Gamma_{MN} d\theta (dX^M dX^N + dX^M \bar{\theta}\Gamma^N d\theta + \frac{1}{3}\bar{\theta}\Gamma^M d\theta \bar{\theta}\Gamma^N d\theta). \tag{28}$$

Varying this, using $\delta_\epsilon \theta = \epsilon$ and $\delta_\epsilon X^M = \bar{\epsilon}\Gamma^M \theta$, one finds that $\mathcal{H}$ is invariant for the choice

$$\begin{aligned} \delta_\epsilon B = & -\frac{1}{2}\bar{\epsilon}\Gamma_{MN}\theta (dX^M dX^N + \frac{2}{3}\bar{\theta}\Gamma^M d\theta dX^N + \frac{1}{15}\bar{\theta}\Gamma^M d\theta \bar{\theta}\Gamma^N d\theta) \\ & -\frac{1}{6}\bar{\epsilon}\Gamma_M \theta \bar{\theta}\Gamma_{MN} d\theta (dX^N + \frac{1}{5}\bar{\theta}\Gamma^N d\theta). \end{aligned} \tag{29}$$

A useful (and standard) identity that has been used in deriving this result is

$$d\bar{\theta}\Gamma^M d\theta d\bar{\theta}\Gamma_{MN} + d\bar{\theta}\Gamma_{MN} d\theta d\bar{\theta}\Gamma^M = 0. \tag{30}$$

The overall normalization of b_3 and $\delta_\epsilon B$ could be scaled arbitrarily (including zero) as far as the present reasoning is concerned. The specific choice that has been made is the one that will be required later. We also note, for future reference, that

$$d\mathcal{H} = -db_3 = -\frac{1}{2}d\bar{\theta}\Gamma_{MN}d\theta\Pi^M\Pi^N = -\frac{1}{2}d\bar{\theta}\psi_5^2 d\theta. \tag{31}$$

where we have introduced the matrix valued one-form

$$\psi_5 = \Gamma_M \Pi_\mu^M d\sigma^\mu. \tag{32}$$

With these choices for $G_{\hat{\mu}\hat{\nu}}$ and $\mathcal{H}$, we can now write down extensions of L_1 and L_3 that have manifest 11d super-Poincaré symmetry:

$$\begin{aligned} L_1 &= -\sqrt{-G}\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2} \\ L_3 &= \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma}, \end{aligned} \tag{33}$$

where z_1 and z_2 are now formed from $\mathcal{H}$ instead of H.

The next step is to construct a supersymmetric extension of L_2. This term is the Wess–Zumino term, which can be represented as the integral of a closed 7-form I_7 over a region that has the 6d world volume M_6 as its boundary. In other words,

$$S_2 = \int_{M_7} I_7 = \int_{M_6} \Omega_6, \tag{34}$$

where $I_7 = d\Omega_6$ and $M_6 = \partial M_7$. The appropriate expression for I_7 that reproduces L_2 of the purely bosonic theory is

$$I_7^{(B)} = -\frac{1}{2}HdH = \frac{1}{2}H\partial_5 H d\sigma^5. \tag{35}$$

To understand this properly, there is a point that needs to be stressed. Namely, in adding a formal 7th dimension, the extra dimension is required to enter symmetrically with the first five. There continues to be one preferred direction, σ^5, that is treated specially. Correspondingly, in writing $M_6 = \partial M_7$, the boundary operator should not act on the σ^5 direction. In other words, M_7 should have no σ^5 = constant faces. It should also be noted that this M theory five-brane theory action has a Wess–Zumino term that survives even for the bosonic truncation in a flat space-time background. However, as we will see in the next subsection, this feature is particular to the non-covariant formulation and is not shared by the PST formulation in which the pieces of the action are arranged somewhat differently.

To complete the construction of L_2 we must now supersymmetrize $I_7^{(B)}$. The term $\frac{1}{2}\mathcal{H}\partial_5\mathcal{H}d\sigma^5$ achieves this, of course, but it is no longer closed. Additional terms should be added such that $dI_7 = 0$, up to a total derivative in the σ^5 direction. The result that we find is

$$I_7 = \frac{1}{2}\mathcal{H}\partial_5\mathcal{H}d\sigma^5 - \frac{1}{2}\mathcal{H}d\bar{\theta}\psi^2 d\theta - \frac{1}{120}d\bar{\theta}\psi^5 d\theta, \tag{36}$$

where

$$\psi = \Gamma_M \Pi^M_{\hat{\mu}} d\sigma^{\hat{\mu}} = \psi_5 + \Gamma_M \Pi^M_5 d\sigma^5. \tag{37}$$

When interpreting the 4-form $d\theta\psi^2 d\theta$ and the 7-form $d\theta\psi^5 d\theta$ it must be understood that one of the derivatives is required to be in the σ^5 direction. The proof that dI_7 is a total σ^5 derivative is reasonably straightforward using the identity (30) as well as

$$\frac{1}{6}(d\bar{\theta}\Gamma_{MNPQR}d\theta d\bar{\theta}\Gamma^R + d\bar{\theta}\Gamma^R d\theta d\bar{\theta}\Gamma_{MNPQR}) = d\bar{\theta}\Gamma_{[MN}d\theta d\bar{\theta}\Gamma_{PQ]}. \tag{38}$$

Since I_7 is manifestly supersymmetric, it is guaranteed that Ω_6 is invariant up to a total derivative under a supersymmetry transformation. For most purposes an explicit formula for L_2 is not required. Here we will simply report that

$$L_2 = -\frac{1}{4}\tilde{H}^{\mu\nu}(\partial_5 B_{\mu\nu} - 2b_{\mu\nu}) + \text{ terms indep. of } B, \tag{39}$$

where $b_2 = \frac{1}{2}b_{\mu\nu}d\sigma^\mu d\sigma^\nu$ is given by[4]

$$\begin{aligned} b_2 \;=\; & -\frac{1}{2}\bar{\theta}\Gamma_{MN}\partial_5\theta(dX^M dX^N + dX^M\bar{\theta}\Gamma^N d\theta + \frac{1}{3}d\bar{\theta}\Gamma^M d\theta d\bar{\theta}\Gamma^N d\theta) \\ & + \frac{1}{2}\bar{\theta}\Gamma_{MN}d\theta(2dX^M\partial_5 X^N - \partial_5 X^M\bar{\theta}\Gamma^N d\theta - dX^M\bar{\theta}\Gamma^N\partial_5\theta - \frac{2}{3}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N\partial_5\theta). \end{aligned} \tag{40}$$

Knowing this much of L_2 is sufficient to obtain the $B_{\mu\nu}$ equation of motion.

4 General Coordinate Invariance

We should now check whether the general coordinate invariance of the bosonic theory in sect. 2.1 continues to hold after adding terms depending on θ in the way that we have described. As in the bosonic case, general coordinate invariance in five directions is manifest, so only the transformation in the σ^5 direction needs to be checked. The coordinates X^M and θ transform as scalars, *i.e.*,

$$\delta_\xi X^M = \xi\partial_5 X^M \quad \text{and} \quad \delta_\xi\theta = \xi\partial_5\theta, \tag{41}$$

[4]This expression is equal to $b_{\mu\nu5}$, where $b_{\hat{\mu}\hat{\nu}\hat{\rho}}$ is the covariant extension of the expression given in eq. (28).

which implies that $G_{\hat{\mu}\hat{\nu}}$ transforms as in eq. (5). To specify the proper transformation law for $B_{\mu\nu}$, we should first examine its equation of motion. Using eq. (39), this is

$$\epsilon^{\mu\nu\rho\lambda\sigma}\partial_\rho\left(K_{\lambda\sigma} - \partial_5 B_{\lambda\sigma} + b_{\lambda\sigma}\right) = 0. \tag{42}$$

The formula for $K_{\mu\nu}$ is as given in eqs. (7) and (8), except that now L_1 and L_3 of the supersymmetrized theory should be used. This simply amounts to replacing H by $\mathcal{H}$ and using the supersymmetric expression for $G_{\hat{\mu}\hat{\nu}}$. By the reasoning explained in ref. [31], the B equation of motion suggests that the appropriate transformation formula, generalizing eq. (6), is

$$\delta_\xi B_{\mu\nu} = \xi(K_{\mu\nu} + b_{\mu\nu}). \tag{43}$$

To determine $\delta_\xi \mathcal{H}$, one first computes that

$$\delta_\xi b_3 = \xi\partial_5 b_3 + b_2 d\xi. \tag{44}$$

It follows that

$$\delta_\xi \mathcal{H} = d(\delta_\xi B) - \xi\partial_5 b_3 - b_2 d\xi = d(\xi K) - \xi Z_3, \tag{45}$$

where

$$Z_3 = \partial_5 b_3 - db_2. \tag{46}$$

This can be made manifestly supersymmetric by noting that

$$Z_3 d\sigma^5 = (\partial_5 b_3 - db_2)d\sigma^5 = -\frac{1}{2}d\bar{\theta}\psi^2 d\theta. \tag{47}$$

The 4-form on the right-hand side of this equation is required to contain one σ^5 derivative.

The important point is that the Z_3 term in $\delta_\xi \mathcal{H}$ has no counterpart in the bosonic theory, so general coordinate invariance of the supersymmetric theory is not an immediate consequence of the corresponding symmetry of the bosonic theory. Let us examine next the part of $\delta_\xi(L_1 + L_3)$ that arises from varying $\mathcal{H}$, but not G. It is

$$\delta_\xi \tilde{\mathcal{H}}^{\mu\nu}\frac{\partial(L_1 + L_3)}{\partial\tilde{\mathcal{H}}^{\mu\nu}} = \frac{1}{2}\delta_\xi\tilde{\mathcal{H}}^{\mu\nu}K_{\mu\nu}. \tag{48}$$

This is conveniently characterized by the 5-form

$$(d(\xi K) - \xi Z_3)K \sim -\xi K(dK + Z_3), \tag{49}$$

where $\sim$ means that a total derivative has been dropped.

Consider now the ξ transformation of L_2. A portion of L_2 was given in eq. (39). Representing this as a 5-form and using

$$\delta_\xi b_2 = \partial_5(\xi b_2), \tag{50}$$

one obtains

$$\begin{aligned} \delta_\xi L_2 &= -(\partial_5 B - b_2) d(\xi(K + b_2)) + H\partial_5(\xi b_2) + \ldots \\ &\sim \xi K(\partial_5 \mathcal{H} + Z_3) + \frac{1}{2} b_2^2 d\xi + \ldots \end{aligned} \tag{51}$$

where the dots are the contribution from varying the H independent terms in L_2. The $\ldots$ terms precisely cancel the b_2^2 term, leaving

$$\delta_\xi L_2 \sim \xi K(\partial_5 \mathcal{H} + Z_3). \tag{52}$$

The demonstration that the $\ldots$ terms contribute $-\frac{1}{2} b_2^2 d\xi$ can be made as follows. The first two terms in eq. (36) contribute the non-H pieces

$$\frac{1}{2} b_3 \partial_5 b_3 d\sigma^5 + \frac{1}{2} b_3 d\bar{\theta} \psi^2 d\theta, \tag{53}$$

which has a non-trivial ξ transformation, because of the asymmetric way in which the σ^5 direction appears. The variation is easy to compute, and can be expressed as the exterior derivative of $-\frac{1}{2} b_2^2 d\xi$, which implies that this contributes the required variation of L_2.

Combining eq. (52) with eq. (49) leaves

$$\delta_{\mathcal{H}}(L_1 + L_3) + \delta_\xi L_2 \sim \xi K(\partial_5 \mathcal{H} - dK). \tag{54}$$

This must now be combined with the terms arising from varying $G_{\hat{\mu}\hat{\nu}}$ in L_1 and L_3. However, at this point all terms whose structure is peculiar to the supersymmetric theory have cancelled. The rest of the calculation is identical to that for the bosonic theory given in ref. [31] and, therefore, need not be repeated here.

5 Proof of Kappa Symmetry

5.1 Formulation Without Manifest Covariance

As with all other super p-branes of maximally supersymmetric theories, the world-volume theory should have 8 bosonic and 8 fermionic physical degrees of freedom. This requires, in particular, the existence of a local fermionic symmetry (called kappa) that eliminates half of the components of θ. Despite the lack of manifest general coordinate invariance, the analysis of kappa symmetry for the M theory five-brane is very similar to that of other super p-branes. As usual, we require that

$$\delta\bar{\theta} = \bar{\kappa}(1-\gamma), \tag{55}$$

where $\kappa(\sigma)$ is an arbitrary Majorana spinor and γ is a quantity (to be determined) whose square is the unit matrix. This implies that $\frac{1}{2}(1-\gamma)$ is a projection operator, and half of the components of θ can be gauged away. In addition, just as for all other super p-branes, we require that

$$\delta X^M = -\delta\bar{\theta}\Gamma^M\theta, \tag{56}$$

so that

$$\delta\Pi^M_{\hat{\mu}} = -2\delta\bar{\theta}\Gamma^M\partial_{\hat{\mu}}\theta. \tag{57}$$

As in our other work [11], we introduce the induced γ matrix

$$\gamma_{\hat{\mu}} = \Pi^M_{\hat{\mu}}\Gamma_M, \tag{58}$$

which satisfies

$$\{\gamma_{\hat{\mu}}, \gamma_{\hat{\nu}}\} = 2G_{\hat{\mu}\hat{\nu}}. \tag{59}$$

In this notation, the kappa variation of the metric is

$$\delta G_{\hat{\mu}\hat{\nu}} = -2\delta\bar{\theta}(\gamma_{\hat{\mu}}\partial_{\hat{\nu}} + \gamma_{\hat{\nu}}\partial_{\hat{\mu}})\theta. \tag{60}$$

Before we can examine the symmetry of our theory, we must also specify the kappa variation of $B_{\mu\nu}$. This works in a way that is analogous to that of the world-volume gauge field for D-branes. Specifically, for the choice

$$\begin{aligned}\delta B = \frac{1}{2}\delta\bar{\theta}\Gamma_{MN}\theta(dX^M dX^N + \bar{\theta}\Gamma^M d\theta dX^N + \frac{1}{3}\bar{\theta}\Gamma^M d\theta\bar{\theta}\Gamma^N d\theta) \\ +\frac{1}{2}\delta\bar{\theta}\Gamma^M\theta\bar{\theta}\Gamma_{MN}d\theta(dX^N + \frac{1}{3}\bar{\theta}\Gamma^N d\theta),\end{aligned} \tag{61}$$

we find that most of the terms in $\delta\mathcal{H}$ cancel leaving

$$\delta\mathcal{H}_{\mu\nu\rho} = 6\delta\bar{\theta}\gamma_{[\mu\nu}\partial_{\rho]}\theta \tag{62}$$

or, equivalently,

$$\delta\tilde{\mathcal{H}}^{\mu\nu} = \epsilon^{\mu\nu\rho\lambda\sigma}\delta\bar{\theta}\gamma_{\rho\lambda}\partial_\sigma\theta. \tag{63}$$

Since we now have the complete theory and all the field transformations, it is just a matter of computation to check the symmetry.

Before plunging into the details of the calculation, it is helpful to sketch the general strategy that will be employed. It turns out to be convenient to consider L_2 and L_3 together and to write their kappa variation in the form

$$\delta(L_2 + L_3) = \frac{1}{2}\delta\bar{\theta}T^{\hat{\mu}}\partial_{\hat{\mu}}\theta. \tag{64}$$

The variation of L_1 is represented in a similar manner:

$$\delta L_1 = -\frac{1}{2L_1}\delta\bar{\theta}U^{\hat{\mu}}\partial_{\hat{\mu}}\theta. \tag{65}$$

Then, in order that $\delta\bar{\theta} = \bar{\kappa}(1-\gamma)$ should be a symmetry, we require that altogether

$$\delta(L_1 + L_2 + L_3) = \frac{1}{2}\delta\bar{\theta}(1+\gamma)T^{\hat{\mu}}\partial_{\hat{\mu}}\theta, \tag{66}$$

which is achieved if

$$U^{\hat{\mu}} = \rho T^{\hat{\mu}}, \tag{67}$$

where

$$\rho = -\gamma L_1 = \gamma\sqrt{-G}\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2}. \tag{68}$$

This implies that

$$\rho^2 = -G(1 + z_1 + \frac{1}{2}z_1^2 - z_2). \tag{69}$$

We must vary the Lagrangian to find $T^{\hat{\mu}}$ and $U^{\hat{\mu}}$, and then determine ρ with the proper square and show that $U^{\hat{\mu}} = \rho T^{\hat{\mu}}$. This is all straightforward, but it needs to be done carefully.

Since the σ^5 direction appears asymmetrically in the Lagrangian, the analysis of $U^{\hat{\mu}} = \rho T^{\hat{\mu}}$ is naturally split into two separate problems, corresponding to $\hat{\mu} = 5$ and $\hat{\mu} \neq 5$. The $\hat{\mu} = 5$ case is the easier of the two, so let us begin with that. We must examine where we can get $\partial_5\theta$'s. The variations of $B_{\mu\nu}$ and $G_{\mu\nu}$ do not give any. Therefore, in varying L_1, the variations of z_1 and z_2 do not contribute. The only contribution comes from

$$\delta\sqrt{-G} = -2\sqrt{-G}\delta\bar{\theta}\gamma^{\hat{\mu}}\partial_{\hat{\mu}}\theta, \tag{70}$$

where, of course, $\gamma^{\hat{\mu}} = G^{\hat{\mu}\hat{\nu}}\gamma_{\hat{\nu}}$. Thus

$$U^5 = -4\rho^2\gamma^5. \tag{71}$$

To determine T^5 we must vary $L_2 + L_3$. Using the identity

$$\delta\left(\frac{G^{5\rho}}{G^{55}}\right) = 2\frac{G_5^{\eta\rho} G^{5\hat{\mu}}}{G^{55}} \delta\bar{\theta}(\gamma_{\hat{\mu}}\partial_\eta + \gamma_\eta \partial_{\hat{\mu}})\theta, \tag{72}$$

the relevant piece of δL_3 is

$$\frac{1}{4}\epsilon_{\mu\nu\rho\lambda\sigma} G_5^{\eta\rho} \delta\bar{\theta}\gamma_\eta \partial_5 \theta \tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma}, \tag{73}$$

which contributes

$$T_2^5 = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda\sigma} G_5^{\eta\rho}\gamma_\eta \tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma} \tag{74}$$

to T^5. (The subscript on T represents the power of $\mathcal{H}$.)

The variation of the Wess–Zumino term S_2 is

$$\delta S_2 = \int (\mathcal{H}\delta\bar{\theta}\psi^2 d\theta - \frac{1}{60}\delta\bar{\theta}\psi^5 d\theta), \tag{75}$$

a result that is obtained by expressing δI_7 as a total differential. This determines $T_0^5 + T_1^5$, with

$$T_0^5 = -\frac{1}{30}\epsilon^{\mu_1\ldots\mu_5}\gamma_{\mu_1\ldots\mu_5} = -4\bar{\gamma}\gamma^5, \tag{76}$$

where we have introduced

$$\bar{\gamma} = \gamma_{012345}, \tag{77}$$

which satisfies $(\bar{\gamma})^2 = -G$. The $\mathcal{H}$ linear term is

$$T_1^5 = -2\tilde{\mathcal{H}}^{\mu\nu}\gamma_{\mu\nu}. \tag{78}$$

Combining these results with

$$U^5 = -4\rho^2\gamma^5 = \rho T^5, \tag{79}$$

we infer that $T^5 = -4\rho\gamma^5$, where

$$\rho = \bar{\gamma} + \frac{1}{2G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}\gamma^5 - \frac{1}{8G^{55}}\epsilon_{\mu\nu\rho\lambda\sigma}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}\gamma^{\sigma 5}. \tag{80}$$

To obtain the $\mathcal{H}^2$ term we have used the identity

$$G_5^{\eta\sigma}\gamma_\eta = \gamma^\sigma - \frac{G^{\sigma 5}}{G^{55}}\gamma^5, \tag{81}$$

from which it follows that

$$G_5^{\eta\sigma}\gamma_\eta\gamma^5 = \gamma^{\sigma 5}. \tag{82}$$

If our reasoning is correct, this expression for ρ should have the square given in eq. (69). This fact is verified in Appendix A.

To complete the proof of kappa symmetry, we must find U^μ and T^μ and show that $U^\mu = \rho T^\mu$. Separating powers of $\mathcal{H}$, as above, the variation of L_2 contributes to T_0^μ and T_1^μ while the variation of L_3 contributes to T_1^μ and T_2^μ. Altogether, we find that

$$\begin{aligned} T_0^\mu &= -4\bar{\gamma}\gamma^\mu \\ T_1^\mu &= -\frac{2}{G^{55}}(G^{5\mu}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho} + 2\tilde{\mathcal{H}}^{\mu\nu}\gamma_\nu\gamma^5) \\ T_2^\mu &= \frac{1}{2G^{55}}\epsilon_{\eta\nu\rho\lambda\sigma}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}(G^{5\mu}G_5^{\eta\zeta}\gamma_\zeta + G_5^{\mu\eta}\gamma^5). \end{aligned} \tag{83}$$

The variation of L_1 determines $U^\mu = \sum_{n=0}^4 U_n^\mu$, where

$$\begin{aligned} U_0^\mu &= 4G\gamma^\mu \\ U_1^\mu &= -\frac{1}{G^{55}}\epsilon^{\mu\nu\rho\lambda\sigma}\gamma_{\lambda\sigma}(G\tilde{\mathcal{H}}G)_{\nu\rho} \\ U_2^\mu &= -\frac{4}{G^{55}}\gamma_\nu(\tilde{\mathcal{H}}G\tilde{\mathcal{H}})^{\mu\nu} - \frac{2}{(G^{55})^2}G^{5\mu}\gamma^5\mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}) \\ U_3^\mu &= \frac{1}{G(G^{55})^2}\epsilon^{\mu\nu\rho\lambda\sigma}\gamma_{\lambda\sigma}\left(\frac{1}{2}(G\tilde{\mathcal{H}}G)_{\nu\rho}\mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}) - (G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G)_{\nu\rho}\right) \\ U_4^\mu &= \frac{4}{G(G^{55})^2}\gamma_\nu\left(\frac{1}{2}(\tilde{\mathcal{H}}G\tilde{\mathcal{H}})^{\mu\nu}\mathrm{tr}(G\mathcal{H}G\mathcal{H}) - (\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}G\tilde{\mathcal{H}})^{\mu\nu}\right) \\ &+ \frac{2}{G(G^{55})^2}\left(G^{\mu5}\gamma^5 - \frac{1}{2}G^{55}\gamma^\mu\right)\left(\frac{1}{2}(\mathrm{tr}(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}}))^2 - \mathrm{tr}(G\mathcal{H}G\mathcal{H}G\mathcal{H}G\mathcal{H})\right). \end{aligned} \tag{84}$$

The demonstration that $U^\mu = \rho T^\mu$ is presented in Appendix B.

In conclusion, we have shown that the theory specified by $L_1 + L_2 + L_3$ has all the desired symmetries: global 11d super–Poincaré symmetry, general coordinate invariance, and local kappa symmetry.

5.2 Supersymmetric Theory in the PST Formulation

The supersymmetric theory that we have just presented can be recast in a manifestly general covariant form, using the PST formalism, just as we did for the bosonic theory in sect. 2.2. In order to keep the notation from being too cumbersome, in this section (and only in this section) indices μ, ν, etc., take six values, (*i.e.*, we drop the hats used until now). Also the label "cov." is dropped. Thus, upon supersymmetrization, eq. (14), for example, becomes

$$M_{\mu\nu} = G_{\mu\nu} + i\frac{G_{\mu\rho}G_{\nu\lambda}}{\sqrt{-G(\partial a)^2}}\tilde{\mathcal{H}}^{\rho\lambda}, \tag{85}$$

where

$$\tilde{\mathcal{H}}^{\rho\lambda} = \frac{1}{6}\epsilon^{\rho\lambda\mu\nu\sigma\tau}\mathcal{H}_{\mu\nu\sigma}\partial_\tau a. \tag{86}$$

Also, $G_{\mu\nu}$ is constructed as in eqs. (25) and (26), and $\mathcal{H} = H - b_3$ is extended to six dimensions. In this notation the supersymmetric theory is given by $L = L_1 + L' + L_{WZ}$, where

$$
\begin{aligned}
L_1 &= -\sqrt{-\det M_{\mu\nu}} \\
L' &= -\frac{1}{4(\partial a)^2}\tilde{\mathcal{H}}^{\mu\nu}\mathcal{H}_{\mu\nu\rho}G^{\rho\lambda}\partial_\lambda a \\
L_{WZ} &= \int \Omega_6 .
\end{aligned}
\tag{87}
$$

L_1 can again be recast in the form

$$
L_1 = -\sqrt{-G}\sqrt{1 + z_1 + \frac{1}{2}z_1^2 - z_2}, \tag{88}
$$

where now z_1 and z_2 are the obvious covariant counterparts of those in eq. (3). The Wess–Zumino term is again characterized by a seven-form $I_7 = d\Omega_6$, where now

$$
I_7 = -\frac{1}{4}\mathcal{H}d\bar{\theta}\psi^2 d\theta - \frac{1}{120}d\bar{\theta}\psi^5 d\theta. \tag{89}
$$

It is easy to check that $dI_7 = 0$ using eqs. (30) and (38). Global ϵ supersymmetry and local reparametrization symmetry are manifest in these formulas. Note that neither the metric $G_{\mu\nu}$ nor the scalar field a occur in L_{WZ}.

When one chooses the gauge $a = \sigma^5$ and $B_{\mu 5} = 0$, the Lagrangian given above reduces to the one in sect. 3. The way this happens is somewhat non-trivial. The point is that L' reduces to L_3 and a portion of the non-covariant Wess–Zumino term L_2. Specifically, in the gauge-fixed theory the sum over the index ρ in the formula for L' can be separated into $\rho = 5$ and $\rho \neq 5$ terms. The $\rho \neq 5$ term accounts for L_3 of the gauge-fixed theory, while the $\rho = 5$ term accounts for the $\mathcal{H}^2$ piece of L_2 and a portion of the $\mathcal{H}$ piece. In particular, this accounts for why the coefficient of the $\mathcal{H}$ linear term in eq. (89) differs from that in eq. (36).

The proof of kappa symmetry in the PST formulation works as before (with $\delta a = 0$), so we will not repeat the argument. [5] The covariant extension of eq. (80) is

$$
\rho = \bar{\gamma} + \frac{1}{2(\partial a)^2}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}\gamma^\lambda\partial_\lambda a - \frac{1}{16(\partial a)^2}\epsilon_{\mu\nu\rho\lambda\sigma\tau}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}\gamma^{\sigma\tau}. \tag{90}
$$

The demonstration that $\rho^2 = -\det M_{\mu\nu}$ is essentially the same as in Appendix A. The covariant formula for $T^\mu = T_0^\mu + T_1^\mu + T_2^\mu$ is given by

$$
T_0^\mu = -4\bar{\gamma}\gamma^\mu
$$

[5] Also, D. Sorokin informs us that it will appear soon in a paper by him and collaborators.

$$
\begin{aligned}
T_1^\mu &= -\frac{2}{(\partial a)^2}\tilde{\mathcal{H}}^{\nu\rho}(\gamma_{\nu\rho}G^{\mu\lambda} - 2\delta^\mu_\rho\gamma_\nu\gamma^\lambda)\partial_\lambda a \\
T_2^\mu &= -\frac{1}{(\partial a)^2}\tilde{\mathcal{H}}^{\eta\nu}\mathcal{H}_{\eta\nu\rho}(\gamma^\rho G^{\lambda\mu} + \gamma^\lambda G^{\rho\mu})\partial_\lambda a \\
&\quad +\frac{2}{[(\partial a)^2]^2}\tilde{\mathcal{H}}^{\eta\nu}\mathcal{H}_{\eta\nu\rho}G^{\rho\lambda}\partial_\lambda a\gamma^\sigma\partial_\sigma aG^{\mu\zeta}\partial_\xi a.
\end{aligned}
\tag{91}
$$

In the $B_{\mu 5} = 0$, $a = \sigma^5$ gauge, these expressions reduce to the formulas T^5 and T^μ given in eqs. (74), (76), (78), and (83). The proof of kappa symmetry works essentially the same as before.

6 Double Dimensional Reduction

As is now well-known, when one of the ten spatial dimensions of M theory is a small circle of radius R, the theory can be reinterpreted as Type IIA string theory in ten dimensions with string coupling constant proportional to $R^{3/2}$ [38, 39]. The five-brane of M theory can then give rise to either a five-brane or a four-brane of Type IIA string theory depending on whether or not it wraps around the circular dimension. Here we wish to focus on the case that it does wrap (once) so that one obtains a four-brane. This case is called "double dimensional reduction," because the dimension of the brane and the dimension of the ambient space-time have been reduced by one at the same time. (The first example of this type to be studied was the double dimensional reduction of the M theory two-brane, which gives the Type IIA fundamental string [9].) The known 4-brane of Type IIA string theory is, in fact, a D-brane, which implies that its world-volume theory contains an abelian vector gauge field. However, the five-brane theory that we have constructed contains an antisymmetric tensor gauge field, which remains one even after the reduction. However, as we will show elsewhere [40], the D 4-brane action and the 4-brane with antisymmetric tensor gauge field obtained below, are related by a world-volume duality transformation. This is analogous to the relationship between the M2-brane and the D2-brane [41, 42, 43].

The covariant action for the dual D4-brane in ten dimensions can be obtained from the M theory five-brane action by setting

$$
X^{11} = \sigma^5 \tag{92}
$$

and then dropping all dependence on σ^5, *i.e.*, extracting the zeroth Fourier mode. Doing this gives

$$\begin{aligned} \psi &\rightarrow \psi + C\Gamma_{11} & (93)\\ G_{\mu\nu} &\rightarrow G_{\mu\nu} + C_\mu C_\nu & (94)\\ b_3 &\rightarrow C_3, & (95)\end{aligned}$$

where

$$C_\mu = -\bar{\theta}\Gamma^{11}\partial_\mu\theta \tag{96}$$

is the part of Π^{11}_μ that survives. C and

$$C_3 = b_3 - \frac{1}{2}\bar{\theta}\Gamma_{11}\Gamma_n d\theta\bar{\theta}\Gamma^{11}d\theta(dX^n + \frac{1}{3}\bar{\theta}\Gamma^n d\theta) \tag{97}$$

enter in the D4-brane Wess-Zumino term. In these formulae quantities on the left (right) of the arrow have target space indices summed on 11 (10) values (e.g., $\psi = \Gamma_M \Pi^M$ on the L.H.S., $\psi = \Gamma_m \Pi^m$ on the R.H.S., where $m = 0, 1, \ldots, 9$ and $M = (m, 11)$). Also,

$$\begin{aligned} G = \det G_{\hat{\mu}\hat{\nu}} &\rightarrow G = \det G_{\mu\nu}\\ G_5 = \det G_{\mu\nu} &\rightarrow \det(G_{\mu\nu} + C_\mu C_\nu) = G(1 + C^2), & (98)\end{aligned}$$

where

$$C^2 \equiv G^{\mu\nu}C_\mu C_\nu. \tag{99}$$

One can analyze the double dimensional reduction of the action. A straightforward calculation shows that

$$\det\left(G_{\hat{\mu}\hat{\nu}} + i\frac{G_{\hat{\mu}\rho}G_{\hat{\nu}\lambda}\tilde{\mathcal{H}}^{\rho\lambda}}{\sqrt{-G_5}}\right) \rightarrow \det\left(G_{\mu\nu} + i\frac{G_{\mu\rho}G_{\nu\lambda}\tilde{\mathcal{H}}^{\rho\lambda}}{\sqrt{-G(1+C^2)}} + Y_\mu Y_\nu\right) \tag{100}$$

with

$$Y_\mu \equiv i\frac{G_{\mu\rho}\tilde{\mathcal{H}}^{\rho\lambda}C_\lambda}{\sqrt{-G(1+C^2)}}, \tag{101}$$

which gives the double-dimensionally reduced version of L_1. For L_3 the answer is:

$$L_3 = \frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{G^{5\rho}}{G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma} \rightarrow -\frac{1}{8}\epsilon_{\mu\nu\rho\lambda\sigma}\frac{C^\rho}{1+C^2}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\sigma}. \tag{102}$$

The Wess–Zumino term is given by the reduction

$$I_7 \rightarrow I_6 = -\frac{1}{4!}d\theta\Gamma_{11}\psi^4 d\theta - \mathcal{H}d\theta\Gamma_{11}\psi d\theta. \tag{103}$$

Under double dimensional reduction

$$d\mathcal{H} = -\frac{1}{2}d\bar{\theta}\psi^2 d\theta \to -\frac{1}{2}d\bar{\theta}\psi^2 d\theta + d\bar{\theta}\Gamma_{11}\psi d\theta C, \tag{104}$$

whose supersymmetry variation is

$$\delta_\epsilon d\mathcal{H} \to d\bar{\theta}\Gamma_{11}\psi d\theta\bar{\epsilon}\Gamma^{11}d\theta. \tag{105}$$

From this one can infer that

$$\delta_\epsilon \mathcal{H} \to (\bar{\epsilon}\Gamma^{11}\theta)d\bar{\theta}\Gamma_{11}\psi d\theta + \text{total derivative}. \tag{106}$$

It is an interesting fact that, after the double dimensional reduction, $\mathcal{H}$ is no longer invariant under supersymmetry. We will show below that the formula has a simple interpretation, which ensures that the reduced theory is supersymmetric. The kappa variations of the doubly dimensionally reduced theory can be analyzed in a similar manner. One finds that

$$\delta\mathcal{H} = -\delta\bar{\theta}\psi^2 d\theta \to -\delta\bar{\theta}\psi^2 d\theta + 2\delta\bar{\theta}\Gamma_{11}\psi d\theta C. \tag{107}$$

In order to preserve the gauge choice (92), both the supersymmetry and the κ variations of the 4-brane fields must include compensating σ^5 general coordinate transformations:

$$\begin{aligned}
0 &= \delta_\epsilon X^{11} + \xi_\epsilon^{\hat{\mu}}\partial_{\hat{\mu}}X^{11} = \bar{\epsilon}\Gamma^{11}\theta + \xi_\epsilon \\
&\Rightarrow \xi_\epsilon = -\bar{\epsilon}\Gamma^{11}\theta \\
0 &= \delta X^{11} + \xi_\kappa^{\hat{\mu}}\partial_{\hat{\mu}}X^{11} = -\delta\bar{\theta}\Gamma^{11}\theta + \xi_\kappa \\
&\Rightarrow \xi_\kappa = \delta\bar{\theta}\Gamma^{11}\theta.
\end{aligned} \tag{108}$$

Upon double dimensional reduction the induced general coordinate transformation parameter ξ only appears in the quantities (see eqs. (45) and (46))

$$\delta_\xi \mathcal{H} = d(\xi K) + \xi db_2 \tag{109}$$

and

$$\delta_\xi C_\mu = \partial_\mu \xi. \tag{110}$$

The supersymmetry variations of C and $\mathcal{H}$ are entirely given by the induced σ^5 general coordinate transformation. Therefore supersymmetry of the theory after double dimensional reduction is a consequence of both the supersymmetry and the general coordinate invariance of the original 6d theory. As a consistency check, one can show that eq. (109) with $\xi = \xi_\epsilon$ reproduces eq. (106). Kappa symmetry works similarly:

$$\delta C_\mu = -\delta\bar{\theta}\Gamma^{11}\partial_\mu\theta - \bar{\theta}\Gamma^{11}\partial_\mu\delta\theta = \partial_\mu\xi_\kappa - 2\delta\theta\Gamma^{11}\partial_\mu\theta, \tag{111}$$

where the second term is the remnant of the κ variation of $G_{\mu 5}$. Looking at $\delta(d\mathcal{H})$ we can compute

$$\delta\mathcal{H} = -\delta\bar{\theta}\psi^2 d\theta + 2\delta\bar{\theta}\Gamma_{11}\psi d\theta C - (\delta\bar{\theta}\Gamma^{11}\theta)d\bar{\theta}\Gamma_{11}\psi d\theta + \text{total derivative}, \tag{112}$$

which is reproduced by combining eqs. (107) and (109) for $\xi = \xi_\kappa$.

7 Discussion

This paper has presented the world-volume action of the M theory five-brane in a flat 11d background. The required global and local symmetries have been verified in detail using a formulation in which one world-volume direction is treated differently from the others. The corresponding results in the manifestly covariant PST formulation have also been presented. Although we have not done it, we expect that it would be reasonably straightforward to extend the results to an arbitrary background, as has been done for D-branes in refs. [12, 13]. All the considerations in this paper have been classical, but there are undoubtedly various quantum implications. In fact, it has been suggested recently that certain supersymmetric 6d theories can have non-trivial renormalization group fixed points [44]. Perhaps our five-brane action is of this type.

The five-brane world volume theory has a solitonic solution [30] that describes a finite-tension self-dual string of the type discussed in [45]. We think that it will be very interesting to study this string and its excitation spectrum, which could then be compared to the spectrum conjectured in [46]. It is curious that the five-brane, which itself arises as a soliton of the 11d theory, has its own solitons. Upon double dimensional reduction to the IIA 4-brane, as discussed in sect. 6, the self-dual string can either wrap or not wrap. This reflects the fact that the D4-brane has both point-like and string-like solitons, which are electric-magnetic duals of one another. The point-like solitons can also be viewed as describing bound states of D4-branes and D0-branes with the D0-brane charge representing momentum in the compact dimension. The string-like solitons do not appear to have an analogous interpretation.

Another direction that we think deserves to be explored is how the M5-brane should be described in the background that describes the $E_8 \times E_8$ theory [47]. The 5-brane in such a background will have half as much supersymmetry as we have described, corresponding to $N = 1$ in 10d. More significantly, it should have a soliton solution that describes a "heterotic" self-dual string. The gauge group, whose currents would appear as left-movers, should be E_8 [48, 49]. It would also be interesting to explore how wrapping M5-branes on suitable 2-cycles gives rise to Seiberg–Witten theories in the unwrapped dimensions [50].

Appendix A – Evaluation of ρ^2

This appendix will show that $\rho^2 = -G(1 + z_1 + \frac{1}{2}z_1^2 - z_2)$, where

$$\rho = \bar{\gamma} + \frac{1}{2G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}\gamma^5 - \frac{1}{8G^{55}}\epsilon_{\mu\nu\rho\lambda\sigma}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}\gamma^{\sigma 5}. \tag{113}$$

It is convenient to rewrite ρ_2 (the subscript refers to the order in $\mathcal{H}$) as

$$\rho_2 = \frac{1}{8G_5}\bar{\gamma}\gamma_{\mu\nu\rho\lambda}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\lambda}, \tag{114}$$

where we have used

$$\frac{1}{k!}\epsilon_{\hat{\mu}_1\ldots\hat{\mu}_6}\gamma^{\hat{\mu}_1\ldots\hat{\mu}_k} = \frac{1}{G}(-1)^{\frac{k(k+1)}{2}}\bar{\gamma}\gamma_{\hat{\mu}_{k+1}\ldots\hat{\mu}_6}. \tag{115}$$

The matrix $\bar{\gamma}$ anticommutes with all γ^μ's, so $\{\rho_0, \rho_1\} = 0$ and $[\rho_0, \rho_2] = 0$. Furthermore,

$$\{\rho_1, \rho_2\} \sim [\gamma_{\alpha\beta}, \gamma_{\mu\nu\rho\sigma}]\tilde{\mathcal{H}}^{\alpha\beta}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma} = 0 \tag{116}$$

as the commutator is antisymmetric over six 5-valued indices. Thus,

$$\rho^2 = \rho_0^2 + \rho_1^2 + 2\rho_0\rho_2 + \rho_2^2. \tag{117}$$

We know already that $\rho_0^2 = -G$ and $\rho_0\rho_2 = -\frac{1}{8G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma}\gamma_{\mu\nu\rho\sigma}$. So we need

$$\begin{aligned}
\rho_1^2 &= \frac{1}{4G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma}\gamma_{\mu\nu}\gamma_{\rho\sigma} = \frac{1}{4G^{55}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma}(\gamma_{\mu\nu\rho\sigma} - 2G_{\mu\rho}G_{\nu\sigma}) \\
&= \frac{1}{4G^{55}}[\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\rho\sigma}\gamma_{\mu\nu\rho\sigma} + 2\mathrm{tr}\,(\tilde{\mathcal{H}}^2)],
\end{aligned}$$

where $\mathrm{tr}\,(\tilde{\mathcal{H}}^2)$ represents $\mathrm{tr}\,(G\tilde{\mathcal{H}}G\tilde{\mathcal{H}})$. Thus, $\rho_1^2 + 2\rho_0\rho_2 = -Gz_1$. Finally,

$$\rho_2^2 = -\frac{G}{64G_5^2}\tilde{\mathcal{H}}^{\mu_1\mu_2}\tilde{\mathcal{H}}^{\mu_3\mu_4}\tilde{\mathcal{H}}^{\nu_1\nu_2}\tilde{\mathcal{H}}^{\nu_3\nu_4}\gamma_{\mu_1\mu_2\mu_3\mu_4}\gamma_{\nu_1\nu_2\nu_3\nu_4}.$$

In the multiplication of gamma matrices[6] one can argue that the only terms that contribute after contraction with the $\mathcal{H}$'s are effectively

$$\gamma_{\mu_1\mu_2\mu_3\mu_4}\gamma_{\nu_1\nu_2\nu_3\nu_4} \sim 8G_{\mu_1\nu_1}G_{\mu_2\nu_2}G_{\mu_3\nu_3}G_{\mu_4\nu_4} - 16G_{\mu_1\nu_2}G_{\mu_2\nu_3}G_{\mu_3\nu_4}G_{\mu_4\nu_1},$$

[6]A useful generalisation of the relation $\gamma_{\mu_1\ldots\mu_m}\gamma_\mu = \gamma_{\mu_1\ldots\mu_m\mu} + m\gamma_{[\mu_1\ldots\mu_{m-1}}G_{\mu_m]\mu}$ is

$$\gamma_{\mu_1\ldots\mu_m}\gamma_{\nu_1\ldots\nu_n} = \sum_{k=0}^{min(m,n)} C_k^{mn}\gamma_{\mu_1\ldots\mu_{m-k}\nu_1\ldots\nu_{n-k}}G_{\mu_{m-k+1}\nu_{n-k+1}}\ldots G_{\mu_m\nu_n},$$

where $C_k^{mn} \equiv (-1)^{kn+\frac{k(k+1)}{2}}k!\binom{m}{k}\binom{n}{k}$. The terms in the sum are antisymmetrized over all μ's and ν's separately.

and thus

$$\rho_2^2 = -\frac{G}{4G_5^2}(\frac{1}{2}\mathrm{tr}\,(\tilde{\mathcal{H}}^2)^2 - \mathrm{tr}\,(\tilde{\mathcal{H}}^4)) = -G(\frac{1}{2}z_1^2 - z_2). \tag{118}$$

Collecting all the terms, we obtain the desired relation:

$$\rho^2 = -G(1 + z_1 + \frac{1}{2}z_1^2 - z_2). \tag{119}$$

Appendix B – Evaluation of ρT^μ

We wish to demonstrate that $\rho T^\mu = U^\mu$, where ρ, T^μ, and U^μ are given by eqs. (80), (83), and (84), respectively. The calculation is somewhat messy, so we proceed order by order in $\mathcal{H}$.

To zeroth order, $\rho_0 = \bar{\gamma}$ and $T_0^\mu = -4\bar{\gamma}\gamma^\mu$ give $\rho_0 T_0^\mu = 4G\gamma^\mu = U_0^\mu$. The linear order contribution comes from $(\rho T^\mu)_1 = \rho_1 T_0^\mu + \rho_0 T_1^\mu$, where

$$\rho_1 = \frac{1}{2G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}{}^5, \qquad T_1^\mu = -\frac{2}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}(G^{5\mu}\gamma_{\nu\rho} + 2G^\mu{}_\nu\gamma_\rho{}^5). \tag{120}$$

Since

$$\rho_1 T_0^\mu = -\frac{2}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\gamma_{\nu\rho}{}^5\bar{\gamma}\gamma^\mu = \frac{2}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\bar{\gamma}(\gamma_{\nu\rho}{}^{5\mu} + G^{5\mu}\gamma_{\nu\rho} + 2G^\mu{}_\nu\gamma_\rho{}^5), \tag{121}$$

we obtain

$$(\rho T^\mu)_1 = \frac{2}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\bar{\gamma}\gamma_{\nu\rho}{}^{5\mu} = -\frac{1}{G^{55}}\epsilon^{\nu\rho\lambda\sigma\mu}\gamma_{\nu\rho}(G\tilde{\mathcal{H}}G)_{\lambda\sigma}, \tag{122}$$

where eq. (115) has been used in obtaining the second equality. Thus, $(\rho T^\mu)_1 = U_1^\mu$.

The higher-order calculations somewhat simplify if one rewrites ρ_2 as

$$\rho_2 = \frac{1}{8G_5}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}\bar{\gamma}\gamma_{\nu\rho\lambda\sigma} \tag{123}$$

and T_2^μ as

$$\begin{aligned} T_2^\mu &= \frac{1}{2G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}\epsilon_{\eta\nu\rho\lambda\sigma}(G^{\mu 5}G_5^{\eta\zeta}\gamma_\zeta + G_5^{\mu\eta}\gamma^5) && (124)\\ &= \frac{1}{2G_5}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}\bar{\gamma}(\gamma_{\nu\rho\lambda\sigma}{}^\mu - 2\frac{G^{\mu 5}}{G^{55}}\gamma_{\nu\rho\lambda\sigma}{}^5), && (125)\end{aligned}$$

using eqs. (81) and (82). In quadratic order,

$$(\rho T^\mu)_2 = \rho_0 T_2^\mu + \rho_1 T_1^\mu + \rho_2 T_0^\mu. \tag{126}$$

If we factor out $(\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma})$ as a common factor in all terms,

$$
\begin{aligned}
\rho_0 T_2^\mu &\sim -\frac{1}{2G^{55}}({\gamma_{\nu\rho\lambda\sigma}}^\mu - \frac{2G^{5\mu}}{G^{55}}{\gamma_{\nu\rho\lambda\sigma}}^5) \\
\rho_1 T_1^\mu &\sim -\frac{1}{(G^{55})^2}{\gamma_{\nu\rho}}^5[G^{5\mu}\gamma_{\lambda\sigma} + 2{G^\mu}_\lambda {\gamma_\sigma}^5] = \\
& -\frac{1}{(G^{55})^2}[G^{5\mu}({\gamma_{\nu\rho\lambda\sigma}}^5 - 2G_{\lambda\nu}G_{\sigma\rho}\gamma^5) - 2G^{55}{G^\mu}_\lambda(\gamma_{\nu\rho\sigma} + 2\gamma_\nu G_{\sigma\rho})] \\
\rho_2 T_0^\mu &\sim -\frac{1}{2G_5}\bar{\gamma}\gamma_{\nu\rho\lambda\sigma}\bar{\gamma}\gamma^\mu = \frac{1}{2G^{55}}({\gamma_{\nu\rho\lambda\sigma}}^\mu + 4\gamma_{\nu\rho\lambda}{G_\sigma}^\mu).
\end{aligned}
$$

Combining these contributions and reinstating $(\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma})$ gives

$$
\begin{aligned}
(\rho T^\mu)_2 &= \frac{1}{G^{55}}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma}[2\frac{G^{5\mu}}{G^{55}}G_{\lambda\nu}G_{\sigma\rho}\gamma^5 + 4{G^\mu}_\lambda\gamma_\nu G_{\sigma\rho}] = \qquad (127) \\
&= -\frac{2}{G^{55}}[\frac{G^{5\mu}}{G^{55}}\mathrm{tr}\,(\tilde{\mathcal{H}}^2)\gamma^5 + 2(\tilde{\mathcal{H}}^2)^{\mu\nu}\gamma_\nu] \\
&= U_2^\mu. \qquad (128)
\end{aligned}
$$

At cubic order in $\mathcal{H}$,

$$
(\rho T^\mu)_3 = \rho_1 T_2^\mu + \rho_2 T_1^\mu. \qquad (129)
$$

Let the common factor to be $(\tilde{\mathcal{H}}^{\alpha\beta}\tilde{\mathcal{H}}^{\nu\rho}\tilde{\mathcal{H}}^{\lambda\sigma})$. Since,

$$
\gamma^5({\gamma_{\nu\rho\lambda\sigma}}^\mu - 2\frac{G^{\mu 5}}{G^{55}}{\gamma_{\nu\rho\lambda\sigma}}^5) = -{\gamma_{\nu\rho\lambda\sigma}}^\mu\gamma^5, \qquad (130)
$$

we get

$$
\begin{aligned}
\rho_1 T_2^\mu &\sim \frac{1}{4G^{55}G_5}\bar{\gamma}\gamma_{\alpha\beta}{\gamma_{\nu\rho\lambda\sigma}}^\mu\gamma^5 \\
&\sim \frac{2}{G^{55}G_5}\bar{\gamma}(-\gamma_{\nu\rho\lambda}G_{\alpha\sigma}\delta^\mu_\beta + \frac{1}{2}{\gamma_{\nu\rho}}^\mu G_{\alpha\sigma}G_{\beta\lambda} - {\gamma_{\nu\lambda}}^\mu G_{\alpha\sigma}G_{\beta\rho})\gamma^5.
\end{aligned}
$$

The second term in eq. (129) can also be simplified:

$$
\begin{aligned}
\rho_2 T_1^\mu &\sim -\frac{1}{4G^{55}G_5}\bar{\gamma}\gamma_{\nu\rho\lambda\sigma}(G^{\mu 5}\gamma_{\alpha\beta} + 2{G^\mu}_\beta\gamma_\alpha\gamma^5) \\
&\sim -\frac{2}{G^{55}G_5}\bar{\gamma}[G^{\mu 5}(-\frac{1}{2}\gamma_{\nu\rho}G_{\alpha\lambda}G_{\beta\sigma} + \gamma_{\nu\lambda}G_{\alpha\rho}G_{\beta\sigma}) - \gamma_{\nu\rho\lambda}G_{\alpha\sigma}\delta^\mu_\beta\gamma^5].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(\rho T^\mu)_3 &= \frac{2}{G^{55}G_5}\bar{\gamma}[{\gamma_{\nu\rho}}^\mu\gamma^5 - G^{\mu 5}\gamma_{\nu\rho}][\frac{1}{2}\tilde{\mathcal{H}}^{\nu\rho}\mathrm{tr}\,(\tilde{\mathcal{H}}^2) - (\tilde{\mathcal{H}}^3)^{\nu\rho}] \\
&= \frac{1}{G^{55}G_5}\epsilon^{\alpha\beta\nu\rho\mu}\gamma_{\alpha\beta}[\frac{1}{2}\tilde{\mathcal{H}}\mathrm{tr}\,(\tilde{\mathcal{H}}^2) - \tilde{\mathcal{H}}^3]_{\nu\rho} \\
&= U_3^\mu. \qquad (131)
\end{aligned}
$$

Finally, in the quartic order,

$$\rho_2 T_2^\mu = -\frac{G}{16G_5^2}\tilde{\mathcal{H}}^{\mu_1\mu_2}\tilde{\mathcal{H}}^{\mu_3\mu_4}\tilde{\mathcal{H}}^{\nu_1\nu_2}\tilde{\mathcal{H}}^{\nu_3\nu_4}\gamma_{\mu_1\mu_2\mu_3\mu_4}({\gamma_{\nu_1\nu_2\nu_3\nu_4}}^\mu - 2\frac{G^{\mu 5}}{G^{55}}{\gamma_{\nu_1\nu_2\nu_3\nu_4}}^5). \tag{132}$$

The relevant contribution of γ's in this case is

$$\begin{aligned}\gamma_{\mu_1\mu_2\mu_3\mu_4}{\gamma_{\nu_1\nu_2\nu_3\nu_4}}^\mu \sim\ & [8\gamma^\mu(G_{\mu_1\nu_1}G_{\mu_2\nu_2}G_{\mu_3\nu_3}G_{\mu_4\nu_4} - 2G_{\mu_2\nu_1}G_{\mu_3\nu_2}G_{\mu_4\nu_3}G_{\mu_1\nu_4}) \\ & -32\gamma_{\nu_1}(\delta^\mu_{\mu_1}G_{\mu_2\nu_2}G_{\mu_3\nu_3}G_{\mu_4\nu_4} - 2\delta^\mu_{\mu_2}G_{\mu_3\nu_2}G_{\mu_4\nu_3}G_{\mu_1\nu_4})].\end{aligned}$$

It follows that

$$\begin{aligned}(\rho T^\mu)_4 &= -\frac{G}{(G_5)^2}\{(\gamma^\mu - 2\frac{G^{\mu 5}}{G^{55}}\gamma^5)[\frac{1}{2}(\mathrm{tr}\,(\tilde{\mathcal{H}}^2))^2 - (\mathrm{tr}\,(\tilde{\mathcal{H}}^4)] \\ &\quad -4\gamma_\nu[\frac{1}{2}\mathrm{tr}\,(\tilde{\mathcal{H}}^2)\tilde{\mathcal{H}}^2 - \tilde{\mathcal{H}}^4]^{\mu\nu}\} \\ &= U_4^\mu. \end{aligned} \tag{133}$$

This completes the proof.

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